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Minimal open strings

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Abstract: We study FZZT-branes and open string amplitudes in (p, q) minimal string theory. We focus on the simplest boundary changing operators in two-matrix models, and identify the corresponding operators in worldsheet theory through the comparison of amplitudes. Along the way, we find a novel linear relation among FZZT boundary states in minimal string theory. We also show that the boundary ground ring is realized on physical open string operators in a very simple manner, and discuss its use for perturbative computation of higher open string amplitudes.

Keywords: Matrix Models, D-branes, Conformal Field Models in String Theory.

Contents

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1. Introduction

Non-critical strings propagating in low-dimensional space-time are interesting toy models of strings [1]–[12]. There are very few dynamical degrees of freedom in such models, and the dynamics is heavily constrained by a large symmetry or integrability. Also, it has long been known that these models have dual non-perturbative descriptions in terms of large N matrix models.

Recent developments in string theories have lead us to realize that D-branes are present also in non-critical string theories. Since a big breakthrough was made in the study of Liouville theory on worldsheets with boundary [13]–[17], many earlier results from matrix models have been revisited and combined with the modern ideas [18]–[34]. This has brought us with a much deeper insight into the models. Now that we have a rather precise understanding of D-branes, it is natural to go further to study the dynamics of open strings. In particular it will be interesting to study how much of the open string dynamics is governed by symmetry.

In this note we wish to study some simple open string amplitudes in (p, q) minimal string theory. We will study them from two different frameworks; using the two-matrix model in section 2 and the worldsheet (p, q) minimal model coupled to Liouville theory in section 3. Along the way, we find a curious linear relation among FZZT boundary states in the worldsheet theory of (p, q) minimal string. In section 3.4 we compute the action of boundary ground ring on physical open string operators and discuss its possible application to higher point amplitudes.

2. Matrix model

It is known that the minimal string theories can be formulated as large N matrix integrals. Throughout this paper we will use the two-matrix model. We begin with reviewing the definition and some fundamental results of this model. See [35, 36] for more detail.

Two-matrix model [8]–[11] is an integral over two $N \times N$ Hermitian matrices X, Y:

$$\int dXdY \exp\left[-\frac{N}{g}\mathrm{Tr}\left(V(X)+U(Y)-XY\right)\right]. \tag{2.1}$$

We assume that $V(X)$ and $U(Y)$ are polynomials of degree q and p, the simplest choice for realizing (p,q) critical behavior. Standard Feynman graph expansion allows us to express the partition function as a sum over fishnet diagrams of arbitrary area (number of vertices) and topology. Each diagram is regarded as a two-dimensional Riemann surface painted by two colors 'X' and 'Y'. The contribution from genus h diagram is proportional to N^{2-2h}, so $1/N$ plays the role of bare string coupling.

After using Harish-Chandra-Itzykson-Zuber formula to reduce the integral to that over the eigenvalues, one is lead to consider the set of polynomials $\{\psi_n(x), \tilde{\psi}_n(y)\}$ satisfying

$$\int dxdy e^{-\frac{N}{g}[V(x)+U(y)-xy]}\psi_n(x)\tilde{\psi}_m(y) \;=\; \delta_{nm}. \tag{2.2}$$

The indices n, m represent the degree of the polynomials. The two matrices then turn into operators $\hat{X}$ and $\hat{Y}$ acting on the set of polynomials as multiplications by x or y. The exact partition function of two-matrix model can be expressed in terms of the matrix elements of $\hat{X}$ and $\hat{Y}$.

Spectral curve. A fundamental observable is the resolvent,

$$R_X(x) \equiv \mathrm{Tr}\frac{1}{X-x}, \qquad R_Y(y) \equiv \mathrm{Tr}\frac{1}{Y-y}. \tag{2.3}$$

They carry the important information on the eigenvalue distributions. Classically at $g=0$ each pair of eigenvalues (x_i, y_i) sits on one of the classical saddle points satisfying $y = V'(x), x = U'(y)$. At nonzero g the eigenvalues spread due to repulsive Coulomb force arising from integrating out the off-diagonal matrix elements.

In the planar approximation, the two equations

$$y = V'(x) + \frac{g}{N}R_X(x), \qquad x = U'(y) + \frac{g}{N}R_Y(y) \tag{2.4}$$

are known to give the same equation on (x,y) defining the *spectral curve*. When regarded as a complex curve, its branch structure reflects how the eigenvalues of X, Y are distributed near each saddle point. It is natural to find the true minimum of the classical action and perform perturbative expansion around that ground state. Such classical ground state should correspond to the spectral curve which is a complex curve of genus zero.

Continuum limit. The idea to get continuous worldsheet is to send $N \to \infty$ and $g \to g_c$ in a suitably correlated manner. Going back to the system of orthonormal polynomials, we find the index n can be replaced by a continuous variable $z = gn/N$ at large N. We parametrize the region $z \sim g_c$ by a new variable $t \equiv \varepsilon^{-2}(g_c - z)$, put $N = \varepsilon^{\gamma-2}$ and take $\varepsilon \to 0$. For judiciously choson potentials, we find the operators $\hat{X}$ and $\hat{Y}$, after suitable rescaling, become a pair of differenial operators

$$\begin{aligned}\hat{X} &\sim d^p + u(t)d^{p-2} + \cdots, \\ \hat{Y} &\sim d^q + v(t)d^{q-2} + \cdots,\end{aligned} \qquad \left(d \equiv \frac{d}{dt},\ \gamma = -\frac{2}{p+q-1}\right)$$

satisfying the canonical commutation relation $[\hat{X}, \hat{Y}] = 1$ or *string equation*. It is known that these operators are conveniently expressed in terms of powers L^j of a pseudo-differential operator $L = d + \mathcal{O}(d^{-1})$, whose positive parts L_+^j generate mutually commuting flows by $\frac{\partial}{\partial t_j} L = [L_+^j, L]$. For $Y = L^q$, the solution to the string equation is

$$X \;=\; -\sum_{j=1}^{p}(1+\frac{j}{q})t_{j+q}L_+^j \;=\; -\sum_{j=1}^{p+q}\frac{j}{q}t_j L^{j-q} + \mathcal{O}(d^{-1-q}). \tag{2.5}$$

The string equation allows us to determine all the coefficient functions $(u, v, \cdots)$ and therefore the partition function as functions of couplings $(t_1, t_2, \cdots)$. For $p > q$, the conformal (p, q) minimal string is obtained by turning on only t_{p+q} and t_{p-q}. After fixing the former, the latter plays the role of the cosmological constant.

The resolvents of two-matrix models for (p, q) minimal string were computed in [5]. The spectral curve is given by $y = R_{\hat{X}}(x)$ and $x = R_{\hat{Y}}(y)$ and has a simple parametric expression [11]

$$x \;=\; 2u^{\frac{p}{2}}\cos(\pi\theta/q), \qquad y \;=\; 2u^{\frac{q}{2}}\cos(\pi\theta/p), \qquad pu^q \;=\; (p-q)t_{p-q}. \tag{2.6}$$

Here $\theta \sim \theta + 2pq$ is the uniformizing parameter. Hereafter we set $u = 1$ for convenience. Using Chebyshev polynomials $T_n(\cos\theta) = \cos n\theta$, the spectral curve can be written in an algebraic form

$$E(x, y) \;\equiv\; T_q(x/2) - T_p(y/2) \;=\; 0. \tag{2.7}$$

2.1 Some disk amplitudes

The resolvent $R_{\hat{X}}(x)$ is related via Laplace transform to the operator $\mathrm{Tr}e^{l\hat{X}}$ that creates a macroscopic loop of length l [6, 7]. We define the FZZT boundary condtion in minimal string theory by weighting each macroscopic loop of length l by a factor e^{-lx}, where x is called the boundary cosmological constant. To the leading order in large N, the correlator

$$-\left\langle \mathrm{Tr}\log(\hat{X} - x)\right\rangle \;=\; \int \frac{dl}{l}\left\langle \mathrm{Tr}e^{l(\hat{X}-x)}\right\rangle \tag{2.8}$$

gives the disk partition function. The resolvent is its first x-derivative so that it has one insertion of boundary cosmological operator $\mathcal{B}$ along the loop. Using the uniformization coordinate θ,

$$\left\langle {}^{\theta}[\mathcal{B}]^{\theta}\right\rangle \;=\; \left\langle \mathrm{Tr}\frac{1}{\hat{X} - x(\theta)}\right\rangle \;=\; y(\theta). \tag{2.9}$$

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When there are more than one insertions of $\mathcal{B}$, one may assign different boundary cosmological constants to each boundary segment. Such amplitudes are the simplest amplitudes of open strings stretching between different FZZT-branes. We can compute them by the iterative use of the simple formula

$$\frac{1}{(\hat{X}-x_1)(\hat{X}-x_2)} = \frac{1}{x_1-x_2}\left(\frac{1}{\hat{X}-x_1}-\frac{1}{\hat{X}-x_2}\right).$$

Explicitly, one finds

$$\left\langle {}^{\theta_1}[\mathcal{B}]^{\theta_2}[\mathcal{B}]^{\theta_1}\right\rangle = \left\langle \mathrm{Tr}\frac{1}{(\hat{X}-x_1)(\hat{X}-x_2)}\right\rangle = \frac{y_1-y_2}{x_1-x_2}, \tag{2.10}$$

$$\begin{aligned}\left\langle {}^{\theta_1}[\mathcal{B}]^{\theta_2}[\mathcal{B}]^{\theta_3}[\mathcal{B}]^{\theta_1}\right\rangle &= \left\langle \mathrm{Tr}\frac{1}{(\hat{X}-x_1)(\hat{X}-x_2)(\hat{X}-x_3)}\right\rangle \\ &= \frac{x_1y_2+x_2y_3+x_3y_1-y_1x_2-y_2x_3-y_3x_1}{(x_1-x_2)(x_2-x_3)(x_3-x_1)},\end{aligned} \tag{2.11}$$

where $x_i = x(\theta_i)$, $y_i = y(\theta_i)$. General n-point amplitude becomes

$$\left\langle {}^{\theta_1}[\mathcal{B}]^{\theta_2}\cdots{}^{\theta_n}[\mathcal{B}]^{\theta_1}\right\rangle = \frac{(-)^{\frac{1}{2}n(n-1)}}{\Delta(x_i)}\det\begin{pmatrix}1 & x_1 & \cdots & x_1^{n-2} & y_1\\ \vdots & \vdots & & \vdots & \vdots\\ 1 & x_n & \cdots & x_n^{n-2} & y_n\end{pmatrix}. \tag{2.12}$$

A more non-trivial boundary operator is the one which changes the color of the boundary, which we call $\mathcal{T}$ in the following. The amplitudes of such operators are given by "mixed-trace" correlators, and they have been extensively studied in a recent work by Eynard, et. al. using the loop equations [37]–[41]. The simplest example is the two-point correlator, which in the planar limit is given by [37]

$$\left\langle \mathrm{Tr}\frac{1}{\hat{X}-x}\frac{1}{\hat{Y}-y}\right\rangle = \frac{E(x,y)}{(x-R_{\hat{X}}(y))(y-R_{\hat{Y}}(x))}. \tag{2.13}$$

As a function of θ,θ' it becomes, up to normalization,

$$\left\langle {}^{\theta}[\mathcal{T}]^{\theta'}[\mathcal{T}]^{\theta}\right\rangle = \frac{2\cos\pi\theta-2\cos\pi\theta'}{\{x(\theta)-x(\theta')\}\{y(\theta)-y(\theta')\}}. \tag{2.14}$$

Note that the enumerator can be factorized,

$$2\cos\pi\theta-2\cos\pi\theta' = \prod_{j=0}^{q-1}\{x(\theta)-x(\theta'+2pj)\} = \prod_{j=0}^{p-1}\{y(\theta)-y(\theta'+2qj)\}. \tag{2.15}$$

Disk amplitudes containing more $\mathcal{T}$'s can be computed using the recursion relation of [38].

3. Worldsheet theory

The worldsheet theory of (p,q) minimal string is the product of a Liouville theory with $b=\sqrt{p/q}$ and a (p,q) minimal model. In this section we generalize this and study the product of two Liouville theories with the couplings b and ib [42, 43]. We start with reviewing the Liouville theory in the presence of boundary.

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3.1 Liouville theory with boundary

Liouville theory with coupling b is a theory of a scalar field ϕ with a potential $\mu e^{2b\phi}$. It is a CFT with central charge

$$c \;=\; 1+6Q^2 \qquad (Q=b+b^{-1}). \tag{3.1}$$

Boundary conditions of Liouville theory are classified by [13, 15]. Some of them, called FZZT boundary states, are described by the boundary interaction $\mu_B\mathcal{B}$, where the cosmological operator $\mathcal{B}\equiv\oint e^{b\phi}$ measures the length of the boundary. We parametrize the boundary states by s, in terms of which μ_B is given by

$$\mu_B \;=\; x(s) \;\equiv\; \sqrt{\mu\pi\gamma(b^2)}\times\frac{\Gamma(1-b^2)}{\pi}\cos(\pi b s).$$

In the following we set $\mu\pi\gamma(b^2)=1$ by a suitable constant shift of the Liouville field. The dual boundary cosmological constant $y(s)$ is related to $x(s)$ by $b\leftrightarrow 1/b$ flip.

Boundary operator $B_k=e^{\frac{(Q+k)\phi}{2}}$ has weight $\frac{Q^2-k^2}{4}$ and satisfies reflection relation

$$^{s}[B_k]^{t} \;=\; {}^{s}[B_{-k}]^{t}\times d(k,s,t). \tag{3.2}$$

The coefficient $d(k,s,t)$ is given by

$$\begin{aligned} d(k,s,t) &= \mathbf{G}(-k)\mathbf{G}(k)^{-1}b^{kb-\frac{k}{b}}\mathbf{S}\left(\frac{Q-k+s+t}{2}\right)\times \\ &\quad\times\mathbf{S}\left(\frac{Q-k+s-t}{2}\right)\mathbf{S}\left(\frac{Q-k-s+t}{2}\right)\mathbf{S}\left(\frac{Q-k-s-t}{2}\right). \end{aligned} \tag{3.3}$$

Here the functions $\mathbf{G}(x)$ and $\mathbf{S}(x)=\mathbf{G}(Q-x)/\mathbf{G}(x)$ are the special functions introduced in [13]. They are characterized by the shift equations

$$\begin{aligned} \mathbf{S}(x+b) &= 2\sin(\pi b x)\mathbf{S}(x), & \mathbf{G}(x+b) &= (2\pi)^{-\frac{1}{2}}b^{\frac{1}{2}-bx}\Gamma(bx)\mathbf{G}(x), \\ \mathbf{S}\left(x+\tfrac{1}{b}\right) &= 2\sin(\pi x/b)\mathbf{S}(x), & \mathbf{G}\left(x+\tfrac{1}{b}\right) &= (2\pi)^{-\frac{1}{2}}b^{\frac{x}{b}-\frac{1}{2}}\Gamma(x/b)\mathbf{G}(x). \end{aligned} \tag{3.4}$$

As a special case, we have

$$d\left(b-\frac{1}{b},s,t\right) \;=\; \frac{y(s)-y(t)}{x(s)-x(t)}. \tag{3.5}$$

Degenerate operators. The boundary operators B_k with special k correspond to degenerate representations. They are used to construct the boundary ground ring elements in minimal string theory. The basic ones are $X\equiv e^{-\frac{b\phi}{2}}$ and $Y\equiv e^{-\frac{\phi}{2b}}$. X or Y are known to connect two boundary states whose s labels differ by $\pm b$ or $\pm b^{-1}$, respectively. Their OPEs with general boundary operators read [13],

$$\begin{aligned} {}^{s'}[X(z)]^{s}[B_k(w)]^{t} &= \sum_{\pm} X_{\mp}\,|z-w|^{\frac{b}{2}(Q\pm k)}\cdot{}^{s'}[B_{k\mp b}(w)]^{t}, \\ {}^{s'}[Y(z)]^{s}[B_k(w)]^{t} &= \sum_{\pm} Y_{\mp}\,|z-w|^{\frac{1}{2b}(Q\pm k)}\cdot{}^{s'}[B_{k\mp\frac{1}{b}}(w)]^{t}. \end{aligned} \tag{3.6}$$

The coefficients are given by $X_- = Y_- = 1$ and

$$X_+ = \frac{2b^2}{\pi}\Gamma(-bk-b^2)\Gamma(bk)\sin\pi\biggl(\frac{b(Q+k\pm s+t)}{2}\biggr)\sin\pi\biggl(\frac{b(Q+k\pm s-t)}{2}\biggr) \qquad (s'=s\pm b),$$

$$Y_+ = \frac{2}{\pi b^2}\Gamma\biggl(-\frac{k}{b}-\frac{1}{b^2}\biggr)\Gamma\biggl(\frac{k}{b}\biggr)\sin\pi\biggl(\frac{Q+k\pm s+t}{2b}\biggr)\sin\pi\biggl(\frac{Q+k\pm s-t}{2b}\biggr) \qquad \biggl(s'=s\pm\frac{1}{b}\biggr). \tag{3.7}$$

The second Liouville theory. As the matter theory, we consider the second Liouville theory with coupling ib and the central charge

$$c \;=\; 1+6\tilde{Q}^2 \qquad (\tilde{Q}=ib-ib^{-1}).$$

The product of Liouville theories with couplings b and ib has critical central charge. We put a tilde to every quantity in the second Liouville theory: for example, the boundary operators $\tilde{B}_{ik}$ have weight $\frac{\tilde{Q}^2+k^2}{4}$. The basic degenerate operators are denoted by $\tilde{X}$ and $\tilde{Y}$, and when multiplied on $\tilde{B}_{ik}$ they shift the momentum k by $\pm b$ or $\pm b^{-1}$.

For $b=\sqrt{p/q}$ the second Liouville theory can be reduced to the (p,q) minimal model with finitely many primary fields forming a closed algebra under fusion. Also, in minimal models there are finitely many boundary states (Cardy states) corresponding to special values of the parameter $\tilde{s}$,

$$\tilde{s} \;\in\; \mathbb{K}\equiv\{lb-kb^{-1}\mid 1\le k\le p-1\,,\ 1\le l\le q-1\}. \tag{3.8}$$

Although their property is significantly different from that of FZZT boundary states in Liouville theory with generic b, the OPE formula (3.6), (3.7) should apply to them as well. This is because the OPE coefficients appearing there are essentially the fusion matrix elements, and they depend on the boundary conditions only through their s-parameters.

3.2 FZZT-branes

The FZZT-brane $|s;k,l\rangle$ in minimal string theory is defined as the direct product of a FZZT boundary state in Liouville theory and the (k,l) Cardy state in minimal model. Its Liouville part is characterized by the boundary cosmological constant and its dual,

$$x(s)=2\cos(\pi sb), \qquad y(s)=2\cos(\pi s/b), \tag{3.9}$$

where some unimportant factors has been dropped. Comparison of this with the result from matrix model shows that the spectral curve has an interpretation as the moduli space of FZZT-branes [24]. The uniformization parameters in the two frameworks are related by $\theta=s\sqrt{pq}$.

Apparently, the worldsheet theory has more D-branes than the two-matrix model, since the branes in the latter do not have labels (k,l). A proposal to resolve this mismatch has been proposed in [24]: it has been observed there that the FZZT-brane $|\theta;k,l\rangle$ in minimal

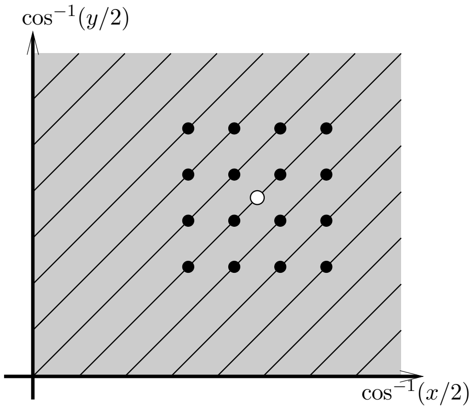

Figure 1: The oblique lines form the spectral curve for the two-matrix model realizing $(p, q) = (8, 7)$ minimal string. The curve covers the x-plane 7 times and y-plane 8 times. The white dot is an FZZT-brane $|\theta; 3, 3\rangle$ which decomposes into nine elementary FZZT-branes described by black dots.

string theory with $(k, l) \neq (1, 1)$ is equivalent to the sum of $(k \times l)$ *elementary branes* $|\theta'\rangle \equiv |\theta'; 1, 1\rangle$,

$$|\theta; k, l\rangle \;\simeq\; \sum_{i,j} |\theta + qj + pi\rangle, \qquad \begin{cases} j \in \{1-k, 3-k, \cdots, k-1\}, \\ i \in \{1-l, 3-l, \cdots, l-1\}. \end{cases} \tag{3.10}$$

This equivalence has been checked in [24] in the sense of BRST cohomology, and derived in [32] using the boundary ground ring. The spectral curve and an example of FZZT-brane is described in figure 1 which nicely encodes the representation theoretic aspect of the (p, q) minimal model.

3.3 Some disk amplitudes

Here we consider some simple disk amplitudes in the generalized minimal string theory with coupling b, whose worldsheet theory is made of two Liouville theories with couplings b and ib. The basic physical boundary operators are boundary tachyons,

$$\mathcal{B}_k \;\equiv\; \mathrm{c} B_k \tilde{B}_{ik}. \tag{3.11}$$

Here c is the reparametrization ghost field. The boundary of the disk is labeled by a pair of parameters $(s, \tilde{s})$. To get (p, q) minimal string theory, we set $b = \sqrt{p/q}$ and restrict k and $\tilde{s}$ to take values in $\mathbb{K}$ of (3.8).

Amplitudes of $\boldsymbol{\mathcal{B}}$. General three-point amplitudes are given by the product of disk three-point functions for the two Liouville theories [17]. Here we focus on the special case where the formula simplifies,

$$\left\langle {}^t[B_k]^s [B_{b-\frac{1}{b}}]^{s'} [B_k]^t \right\rangle \;=\; k \frac{d(k, s, t) - d(k, s', t)}{x(s) - x(s')}. \tag{3.12}$$

From this we get the three-point amplitude

$$\left\langle {}^{(t,\tilde{t})}[\mathcal{B}_k]^{(s,\tilde{s})}[\mathcal{B}_{b-\frac{1}{b}}]^{(s',\tilde{s})}[\mathcal{B}_k]^{(t,\tilde{t})} \right\rangle \;=\; k\frac{d(k,s,t)-d(k,s',t)}{x(s)-x(s')}\cdot \tilde{d}(ik,\tilde{s},\tilde{t}). \tag{3.13}$$

Notice that $\mathcal{B}_{b-\frac{1}{b}} = \mathcal{B}_{(1,1)}$ is nothing but the boundary cosmological operator $\mathcal{B}$. Restricting to (p,q) minimal string theory and setting $k=\tilde{s}=\tilde{t}=b-\frac{1}{b}$, the three-point amplitude becomes

$$\left\langle {}^{(s_1)}[\mathcal{B}]^{(s_2)}[\mathcal{B}]^{(s_3)}[\mathcal{B}]^{(s_1)} \right\rangle \;=\; \frac{p-q}{\sqrt{pq}}\frac{x_1y_2+x_2y_3+x_3y_1-y_1x_2-y_2x_3-y_3x_1}{(x_1-x_2)(x_2-x_3)(x_3-x_1)}, \tag{3.14}$$

where we used $x_i = x(s_i)$, $y_i = y(s_i)$. This is in agreement with the matrix model result (2.11).

In the limit $s' \to s$ the right hand side of (3.13) becomes a derivative with respect to x. We assume that one can integrate it when the operators inserted are all within Seiberg's bound [44], since it would lead to an inconsistency if we could always integrate it [45]. We thus find the two-point amplitude

$$\left\langle {}^{(t,\tilde{t})}[\mathcal{B}_k]^{(s,\tilde{s})}[\mathcal{B}_k]^{(t,\tilde{t})} \right\rangle \;=\; \mathrm{sgn}(\mathrm{Re}k)kd(k,s,t)\tilde{d}(ik,\tilde{s},\tilde{t}). \tag{3.15}$$

Restricting to (p,q) minimal string and $k=(1,1)$, we again find the agreement with matrix model result (2.10),

$$\left\langle {}^{(s_1)}[\mathcal{B}]^{(s_2)}[\mathcal{B}]^{(s_1)} \right\rangle \;=\; \frac{p-q}{\sqrt{pq}}\frac{y_1-y_2}{x_1-x_2}. \tag{3.16}$$

We can integrate further and check that the one-point amplitude agrees with the resolvent in two-matrix model.

Amplitudes of $\mathcal{T}$. Next we consider the general two-point amplitude in (p,q) minimal string.

$$\left\langle {}^{(s';k',l')}[\mathcal{B}_{(n,m)}]^{(s;k,l)}[\mathcal{B}_{(n,m)}]^{(s';k',l')} \right\rangle \;\sim\; \frac{|mp-nq|}{\sqrt{pq}}d(mb-nb^{-1},s,s'). \tag{3.17}$$

The amplitude is non-vanishing only when the representation (n,m) is allowed between two Cardy states (k,l) and (k',l') in minimal model. More explicitly

$$\begin{aligned} |k-k'|+1 \le\; n \;&\le \min(k+k'-1, 2p-k-k'-1), \\ |l-l'|+1 \le\; m \;&\le \min(l+l'-1, 2q-l-l'-1). \end{aligned} \tag{3.18}$$

Using $\theta = s\sqrt{pq}$ and $\theta' = s'\sqrt{pq}$, the amplitude is proportional to

$$\sim \frac{\prod_{j=0}^{n-1}\{y(\theta')-y(\theta+p(1-m)+q(1-n+2j))\}}{\prod_{j=0}^{m-1}\{x(\theta')-x(\theta+q(1-n)+p(1-m+2j))\}}.$$

Comparing this with (2.14) one finds the correspondence

$${}^{\theta}[\mathcal{T}]^{\theta'} \;\sim\; {}^{(\theta;k,l)}[\mathcal{B}_{(p-1,1)}]^{(\theta'-pq;p-k,l)}. \tag{3.19}$$

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Thus we identified the two boundary operators $\mathcal{B}$ and $\mathcal{T}$ in two-matrix model with the boundary operators $\mathcal{B}_{(1,1)}$ and $\mathcal{B}_{(p-1,1)}$ in the worldsheet theory. These operators are both at the corner of Kac table. One of their special properties is that, when fused with any primary field, they produce only one primary.

Interestingly, by translating some four-point amplitudes from two-matrix model into worldsheet theory, one finds that the amplitudes become non-invariant under Liouville reflection of operators (3.2). The standard interpretation for this is that the insertion of four or more operators is enough to deform the theory away from the Liouville background.

New linear relation among D-branes. Note that (3.19) also suggests the equivalence between FZZT-branes

$$|\theta;k,l\rangle \;=\; -\,|\theta-pq;p-k,l\rangle. \tag{3.20}$$

The minus sign is required for the equalities with different (k,l) to be mutually consistent. More interestingly, when these equalities are combined with (3.10), they give rise to simple linear relations among elementary FZZT-branes,

$$0 \;=\; \sum_{j=1}^{p}|\theta+2qj\rangle \;=\; \sum_{j=1}^{q}|\theta+2pj\rangle\,, \tag{3.21}$$

which say that p or q elementary FZZT-branes can disappear into nothing when placed in a suitable manner. These equalities can be checked in the sense of BRST cohomology in the same way as (3.10) was checked.

3.4 Boundary ground ring

The worldsheet theory has boundary operators labeled by (k,l), but the two-matrix model does not seem to have corresponding boundary changing operators. In other matrix models such as height models [3 – 5], there seem to be more boundary operators and we may make a more direct comparison with the worldsheet theory [46]. On the other hand, different boundary operators in worldsheet theory are related by the action of boundary ground ring [47, 23, 32] so that we may well regard them as redundant.

There is a set of physical operators of ghost number zero in minimal string theory which form the *ground ring*. Here we consider the ring of boundary operators. The ring elements $\mathcal{O}_{m,n}$ are constructed from the (m,n) degenerate Liouville operator and the (m,n) operator in minimal model. The ring is generated by the operators $\mathcal{X}=\mathcal{O}_{1,2}$ and $\mathcal{Y}=\mathcal{O}_{2,1}$,

$$\begin{aligned}\mathcal{X} &\equiv \frac{1}{2b^2}(b^{+2}\,\mathrm{bc}+L_{-1}-\tilde{L}_{-1})X\tilde{X},\\ \mathcal{Y} &\equiv \frac{b^2}{2}(b^{-2}\,\mathrm{bc}+L_{-1}-\tilde{L}_{-1})Y\tilde{Y}.\end{aligned} \tag{3.22}$$

Here b,c are reparametrization ghosts. The ring relation is realized linearly on the physical boundary operators $\mathcal{B}_k$. Schematically one has

$$\begin{aligned}\mathcal{X}\mathcal{B}_k &= \sum_{\pm}\mathcal{X}_\pm(k)\mathcal{B}_{k\pm b},\\ \mathcal{Y}\mathcal{B}_k &= \sum_{\pm}\mathcal{Y}_\pm(k)\mathcal{B}_{k\pm\frac{1}{b}}.\end{aligned} \tag{3.23}$$

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The coefficients $\mathcal{X}_\pm, \mathcal{Y}_\pm$ can be computed using the formulae (3.6). Similar formulae hold also for right multiplications. Note that the coefficients depend on the boundary parameters though we will suppress it for notational simplicity. Note also that the boundary parameters s and $\tilde{s}$ have to jump by $\pm b$ or $\pm b^{-1}$ where $\mathcal{X}$ or $\mathcal{Y}$ are inserted.

The linear action of $\mathcal{X}, \mathcal{Y}$ on boundary tachyons satisfies the following. First, the left- and right-multiplications commute for all pairs of operators,

$$(\mathcal{X}\mathcal{B})\mathcal{Y} \;=\; \mathcal{X}(\mathcal{B}\mathcal{Y}), \qquad (\mathcal{X}\mathcal{B})\mathcal{X} \;=\; \mathcal{X}(\mathcal{B}\mathcal{X}), \qquad \text{etc.} \tag{3.24}$$

Also, the multiplications of an $\mathcal{X}$ and a $\mathcal{Y}$ from the same side *anticommute*,

$$\mathcal{X}\mathcal{Y}\mathcal{B} \;=\; -\,\mathcal{Y}\mathcal{X}\mathcal{B}. \tag{3.25}$$

To simplify the formulae that follow, we introduce the notation

$$\begin{aligned} \mathcal{X}_\pm &= {}^{(s\pm b,\tilde{s}-b)}\mathcal{X}^{(s,\tilde{s})}, & \mathcal{Y}_\pm &= {}^{(s\mp\frac{1}{b},\tilde{s}+\frac{1}{b})}\mathcal{Y}^{(s,\tilde{s})}, \\ \bar{\mathcal{X}}_\pm &= {}^{(s\pm b,\tilde{s}+b)}\mathcal{X}^{(s,\tilde{s})}, & \bar{\mathcal{Y}}_\pm &= {}^{(s\mp\frac{1}{b},\tilde{s}-\frac{1}{b})}\mathcal{Y}^{(s,\tilde{s})}. \end{aligned} \tag{3.26}$$

They can be shown to satisfy the algebraic relations

$$\begin{aligned} \bar{\mathcal{X}}_-\mathcal{X}_+ - \bar{\mathcal{X}}_+\mathcal{X}_- &= \sin(\pi b s)\sin(\pi b\tilde{s} - \pi b^2), \\ \mathcal{X}_+\bar{\mathcal{X}}_- - \bar{\mathcal{X}}_+\mathcal{X}_- &= \sin(\pi b s - \pi b^2)\sin(\pi b\tilde{s}), \\ \bar{\mathcal{Y}}_-\mathcal{Y}_+ - \bar{\mathcal{Y}}_+\mathcal{Y}_- &= \sin(\tfrac{\pi s}{b})\sin(\tfrac{\pi\tilde{s}}{b} + \tfrac{\pi}{b^2}), \\ \mathcal{Y}_+\bar{\mathcal{Y}}_- - \bar{\mathcal{Y}}_+\mathcal{Y}_- &= \sin(\tfrac{\pi s}{b} + \tfrac{\pi}{b^2})\sin(\tfrac{\pi\tilde{s}}{b}), \end{aligned} \tag{3.27}$$

and commutation relations

$$\begin{aligned} [\mathcal{X}_+, \bar{\mathcal{X}}_-] &= -\sin\pi b^2 \sin\pi b(\tilde{s} - s), & [\mathcal{Y}_+, \bar{\mathcal{Y}}_-] &= \sin\tfrac{\pi}{b^2}\sin\tfrac{\pi}{b}(\tilde{s} - s), \\ [\mathcal{X}_-, \bar{\mathcal{X}}_+] &= -\sin\pi b^2 \sin\pi b(\tilde{s} + s), & [\mathcal{Y}_-, \bar{\mathcal{Y}}_+] &= \sin\tfrac{\pi}{b^2}\sin\tfrac{\pi}{b}(\tilde{s} + s). \end{aligned} \tag{3.28}$$

All other commutators vanish, i.e.

$$[\mathcal{X}_\pm, \bar{\mathcal{X}}_\pm] = [\mathcal{X}_+, \mathcal{X}_-] = [\bar{\mathcal{X}}_+, \bar{\mathcal{X}}_-] = [\mathcal{Y}_\pm, \bar{\mathcal{Y}}_\pm] = [\mathcal{Y}_+, \mathcal{Y}_-] = [\bar{\mathcal{Y}}_+, \bar{\mathcal{Y}}_-] = 0.$$

Linear relations among D-branes revisited. Thanks to the above simple algebraic relations, we may construct general ring elements as simple products of generators without worrying about the order of multiplication. Let us now consider the (p,q) minimal string theory and restrict k and $\tilde{s}$ to take values in $\mathbb{K}$. Let us introduce

$$\begin{aligned} {}^{(\theta')}[\mathcal{O}_{k,l}]^{(\theta;k,l)} &= {}^{(\theta')}[\mathcal{Y}_-^{k_-}\mathcal{Y}_+^{k_+}\mathcal{X}_-^{l_-}\mathcal{X}_+^{l_+}]^{(\theta;k,l)}, \\ {}^{(\theta;k,l)}[\bar{\mathcal{O}}_{k,l}]^{(\theta')} &= {}^{(\theta;k,l)}[\bar{\mathcal{X}}_+^{l_+}\bar{\mathcal{X}}_-^{l_-}\bar{\mathcal{Y}}_+^{k_+}\bar{\mathcal{Y}}_-^{k_-}]^{(\theta')}, \end{aligned} \tag{3.29}$$

where $k, l, k_\pm, l_\pm, \theta$ and θ' satisfy

$$\theta' = \theta + p(l_+ - l_-) - q(k_+ - k_-), \quad k = k_+ + k_- + 1, \quad l = l_+ + l_- + 1. \tag{3.30}$$

These operators can be used to generalize the relations (3.10) to the branes appearing on boundary segment. The naive application of the formula to an FZZT-brane between two boundary operators would lead to a conflict with Cardy's constraint. The correct way is to put a suitable pair of boundary ground ring elements at the ends of the segment. By repeatedly using the first and third equalities in (3.27), we find

$$]^{(\theta;k,l)}[\;=\; \sum_{\theta'} \frac{]^{(\theta;k,l)}[\bar{\mathcal{O}}_{k,l}]^{(\theta')}[\mathcal{O}_{k,l}]^{(\theta;k,l)}[}{F_{\theta'}(\theta;k,l)}, \tag{3.31}$$

where the function $F_{\theta'}(\theta;k,l)$ is given by

$$\begin{aligned} F_{\theta'}(\theta;k,l) \;=\; & (-1)^{l_+ + k_-} \prod_{j=-l_-}^{l_+} \sin\frac{(\theta+jp)\pi}{q} \prod_{j=-k_-}^{k_+} \sin\frac{(\theta-jq)\pi}{p} \\ & \times \prod_{j=1}^{k_+} \sin\frac{jq\pi}{p} \prod_{j=1}^{k_-} \sin\frac{jq\pi}{p} \prod_{j=1}^{l_+} \sin\frac{jp\pi}{q} \prod_{j=1}^{l_-} \sin\frac{jp\pi}{q}. \end{aligned} \tag{3.32}$$

The formula (3.31) can also be used to relate the three-point amplitudes of general boundary operators to that of three boundary cosmological operators, since

$${}^{(\theta)}[\mathcal{O}_{k,l}\mathcal{B}_{(m,n)}\bar{\mathcal{O}}_{k',l'}]^{(\theta')}$$

should always be proportional to $\mathcal{B}_{(1,1)}$ from Cardy's constraint.

Recursion relations for open string amplitudes. Using the operators $\mathcal{X}, \mathcal{Y}$ one can derive recursion relations among three-point amplitudes. Omitting the dependence on boundary parameters, one has schematically

$$\begin{aligned} 0 &= \langle \mathcal{B}_{k_1}[Q_B, \mathcal{X}]\mathcal{B}_{k_2}\mathcal{B}_{k_3}\rangle \\ &= \langle (\mathcal{B}_{k_1}\mathcal{X})\mathcal{B}_{k_2}\mathcal{B}_{k_3}\rangle - \langle \mathcal{B}_{k_1}(\mathcal{X}\mathcal{B}_{k_2})\mathcal{B}_{k_3}\rangle \\ &= \sum_{\pm} \mathcal{X}_{\pm}(k_1)\langle \mathcal{B}_{k_1\pm b}\mathcal{B}_{k_2}\mathcal{B}_{k_3}\rangle + \sum_{\pm} \mathcal{X}_{\pm}(k_2)\langle \mathcal{B}_{k_1}\mathcal{B}_{k_2\pm b}\mathcal{B}_{k_3}\rangle. \end{aligned} \tag{3.33}$$

Similar recursion relation can be shown to hold also for two-point amplitudes. The idea to get these recursion relations is to rewrite the amplitudes containing Q_B-exact operator into an integral over the boundary of moduli space or a sum over factorized worldsheets. The same arguments can be applied to obtain recursion relations for higher amplitudes.

Concrete recursion relations have been proposed in $c = 1$ string theory by [23] and in minimal string theory by [32], following the argument of [47] that the recursion relations boil down to the higher operator product algebras such as

$$(\mathcal{B}_{k_1}\cdots\mathcal{B}_{k_n}\mathcal{X}\mathcal{B}_{k_{n+1}}\cdots\mathcal{B}_{k_N}) \;\longrightarrow\; \mathcal{B}_{k'}.$$

However, we do not see any obvious reason that the operator products vanish for $N \geq 3$ in the worldsheet theory with nonzero cosmological coupling, though it was assumed in many literature.

In a recent paper [45] the recursion relation for four-point amplitudes in $c = 1$ theory has been solved and shown to reproduce the matrix model result. It will be important to understand better the symmetry structure of minimal string theory by making use of the boundary ground ring relations in worldsheet theory and the loop equations in two-matrix model.

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